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THE SPACE CONCEPT

BY

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Descriptive

1·1. When speaking of the Number-Concept we showed that it was based on the *possibility* of recognising a homogeneity as well as a heterogeneity in the elements of our experience; that the homogeneity and the heterogeneity themselves were the basis of our abstractions. We referred also to "Order" and its connection with the number-system. Unity was taken as an "a priori" concept, and found to be the elemental concept of the other numbers. When we attempt to deal with Space we shall find the very same fundamental requirements, viz., the possibility of recognising a unity underlying our spatial elements as well as a diversity among them, and the possibility of order in the elements.

1·2. Before entering upon our subject it will be advisable to indicate the plan of our treatment. Firstly, it may be noted that our method is not controversial, nor is it historical. Our aim is to set forth *logically* how our concept of Space is built up, as it were, from simpler concepts. This we shall do by making use of all the results of that science of Space which in particular is called Geometry. For, as we said in our previous article, complex concepts will submit to analysis only after a great deal of their use. The process of analysis will help us further in the understanding of Nature in general. It will be necessary, of course, to begin by defining several of the terms we employ; for, of all the confusion that has been introduced by the vague meanings assigned to expressions there has been no greater than that so often met with in philosophical treatises. Our discussion *will* finally lead us into the domains of Philosophy, whose bounds we should fain enter with cautious steps. After fixing our terms we shall follow up the growth of our concept through that logically prior part of Geometry, called Descriptive Geometry, then through Metrical Geometry—both Euclidean and non-Euclidean—and finally through that mathematically generic notion of space as a Manifold, leading up to the concrete Space-Time continuum.

1·3. To begin with, let us observe that fundamental "definitions" in mathematics are purely nominal; that is, they are only the giving of names to concepts. Hence they are arbitrary. As we know, the criterion of the appositeness of any definition is that it states the logically essential part of the concept. In mathematics, however, this is impossible; for there are several

"necessary and sufficient" consequences to a statement, and any one of them may be taken as the definition. It is for this reason that in our subject any fundamental analysis by means of definitions will lead to a vicious circle. The problem is like finding out which of those sitting round a circular table can properly be said to hold the first seat. Our choice will be entirely arbitrary. So it is with the fundamental definitions of mathematics.

1·4. Geometry deals not with space but with spatial figures, or figures in space. When we have done with these and attempt to deal with "space", we break into the bounds of metaphysics. Before giving definitions to spatial figures, we must know firstly whether these figures are *possible*. We *assume* the possibility from the fact that they are the subject-matter of our science.

1·5. This is the question we set before ourselves: What are the simplest concepts which are involved in our complex concepts of spatial figures? We must therefore investigate the elements to which we can reduce our spatial figures, and the "adjectives of space", or those peculiar properties of the elements from which all the properties of spatial figures can be derived. The properties of the elements should be logically independent, and the fewest possible; moreover, they should be simple statements whose import is clearly understood. These are called the "axioms", which may therefore be considered as the (logically) "a priori" properties of spatial elements. We take "a priori" in the logical sense, to mean that (statement) without which no knowledge can be got. These axioms may be worded in different ways, and for choice of two statements. A and B, if B follows necessarily from A and A necessarily from B, and the other implications of both are the same, then we may choose either A or B as the axiom; but we must recognise that the fundamental spatial adjective given to us by either is the same. All the fundamental spatial adjectives connected with our elements will together go to make up our complex concept of a spatial figure.

1·6. Another word that needs to be used in a precise sense is "empirical". Any property of spatial figures, *which we cannot arrive at by reasoning from our axioms, but for which we depend on sense-experience*, we shall call an "empirical property", the statement of it an "empirical statement", and the "truth" that is attached to it an "empirical truth". The meaning of "truth" itself we shall leave for consideration to the very end of our philosophical lucubrations. This, be it observed, is not the same as calling everything empirical which we know through the sensations. For, certainly, all our knowledge comes firstly through sensation, according to the scholastic adage: "nihil est in intellectu quod non prius fuerit in sensu". The axioms are abstractions from intellectual concepts, abstractions indeed from abstractions, and therefore they less merit the epithet "empirical" than anything else. Starting from sensations and perceptions, we abstract our complex notions, and on analysing these we abstract still further to get the axioms.

1·7. Another word that is much misused is "subjective". We should definitely distinguish this word from "ideal" and "abstract" on the one hand

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and "a priori" on the other. No objection will be taken to the use of this word as signifying that particular aspect of a thing which would be perceived by a particular being knowing it in his particular way. However, the word, not being of much use in our present discussion, it will be advisably avoided. For this reason we shall not go further into the question of the subjectivity or the objectivity of the things we deal with in Geometry. For mathematical purposes all minds are alike, and all concepts of the same thing alike; else our investigations would be abortive, if not meaningless. This is the necessary condition for the *possibility* of the discovery of axioms, and of the analysis of complex concepts. We do not speak of concepts as yours, or mine, or Tom's, or Dick's, or Harry's, but as belonging to the human race, or to beings with our Laws of Thought.

1·8. Pure Geometry can be distinguished into two branches, Descriptive Geometry proper and Metrical Geometry; and the latter can be partitioned off into Euclidean and non-Euclidean. Descriptive Geometry deals with spatial figures without making use of the concept of Quantity. The axioms of Descriptive Geometry are presupposed in Metrical Geometry. They are not different geometries: the former is the development from certain fundamental axioms; when the axioms involving quantity are added further development becomes possible; some more axioms are needed for Euclidean Geometry, which distinguish it from non-Euclidean Geometry.

1·9. We have seen in the previous article how we abstracted from matter the concepts of unity and multiplicity, and of order; and we have shown that these were the "a priori" concepts for the number-system; and how the system of integral positive numbers was extended on account of an algebraical necessity. In Geometry we deal with abstractions away from such properties of matter as are capable of affecting our senses, from all causal properties; that is, of acting physically on or being acted upon by some material body; and as much as possible we abstract from time also.

1·10. Let us look firstly for our basis of unity and diversity in the spatial elements. The elements of Descriptive Geometry are: the plane, the straight-line, and the point. These we must consider as strictly inter-related concepts. We cannot have a concept of the point without that of the straight-line, and *vice versa*, or of the straight-line without that of the plane, and *vice versa*. The definition of one without an implication of the other would not be possible. The logically vicious circle becomes only apparent, if we do not separate our notions but treat them as analytically essential and inseparable parts of a more complete concept.

1·11. Consider points as elements of spatial figures. Whatever they are they must have a certain unity and a diversity as well. The same applies to straight lines, whatever they are, and to planes also. Without this assumption we cannot get any further. It is an "a priori" assumption. Moreover, the unity and the diversity, the basis of identification and of differentiation, should

be in one and the same logical predicament that we attribute to them. When dealing with Descriptive Geometry, the basis of the identification of two points is taken as the straight line. The points A and B are identical or alike in that they have a common property, the straight line, which we call the base l; A and B are different, or unlike, in as much as all the properties of A are not the same as those of B: we can have a point C which has a property in common with A, the base l', and a property in common with B, the base l'', but these themselves are different, so that l, l', and l'', are not one and the same property. Thus we have the basis required.

1·12. Correlative remarks hold for the straight line. Consider straight lines as elements of spatial figures. The basis for their identification and differentiation are descriptive properties, called points. Two straight lines, a and b, are the same or alike, in that they have a common property, the point, which we call the vertex V; a and b are different, or unlike, in as much as all the properties of a are not the same as those of b: we can have a straight line c, whose property in common with a is not one and the same as that which it has in common with b.

1·13. Thus the straight line is an element as well as a descriptive property, and the point is also an element as well as a descriptive property. The points A and B "determine" a straight line l, and two straight lines a and b determine a point V.

1·14. It is an easy step then to the next aspect of the straight line and of the point. The points themselves that have the common descriptive property l we can call the straight line l; and the straight lines that have the common descriptive property V we can call the point V.

1·15. Hence we have already three aspects of the straight line and three of the point: the straight line as an element the descriptive properties of which are points; as itself a descriptive property of elemental points; and as the aggregate of all the elemental points with this descriptive property in common. Two eleemental points determine their common descriptive property, and two elemental straight lines determine their common descriptive property. Again the point can be considered as an element the descriptive properties of which are straight lines; as itself a descriptive property of elemental straight lines; and as the aggregate of all the straight lines with this descriptive property in common. Two elemental straight lines determine their common descriptive property, and two elemental points determine their common descriptive property. Thus considering points as spatial elements—we have called them elemental points—straight lines may be reckoned as their properties, we have called them descriptive straight lines; and considering straight lines as spatial elements—we have called them elemental straight lines—points may be reckoned as their properties, we have called them descriptive points. The straight line may be considered as the aggregate of elemental points having a property in common, and the point as the aggregate of elemental straight lines having one property in common. Thus points may be said to lie on a straight

line and straight lines to pass through a point. Two straight lines determine a point in a double sense: there is one elemental point with two different descriptive properties, and there is one descriptive point common to two elemental straight lines. So also two points may be said to determine a straight line.

1·16. In the same way the plane and the straight line are strictly correlative ideas. Considering the plane as a spatial element the straight line is the unifying and the distinguishing property of it. Two planes are identical or alike in that both have one common property, the descriptive straight line; they are unlike when the property which each has in common with some third plane is not one and the same straight line. Two elemental straight lines are like when they both have one common property the descriptive plane; and they are unlike when the property which each has in common with a third straight line is not one and the same plane.

1·17. Again we may consider the straight line as the aggregate itself of all the elemental planes which have that straight line as their descriptive property, so that the straight line is again viewed under three aspects; and we may consider the plane as the aggregate itself of all the elemental straight lines which have that plane as their common descriptive property, so that the plane also can be considered under three aspects: viz., as an element, as the descriptive property of the straight line, and as the aggregate of elemental straight lines. But since the straight line itself is an aggregate of elemental points, the plane also can be considered as an aggregate of points. Any two elemental points have a common property, the straight line as descriptive, and they have also the properties of all the descriptive planes which that straight line as elemental has. Again since two elemental planes determine a straight line, and the straight line lies in both the planes, it FOLLOWS that, if two points of a straight line lie in a plane, all its points lie in the plane. Therefore also two straight lines will determine a plane when they have a point in common, and two straight lines will determine a point when they have a plane in common; a straight line and a point not on it will determine a plane; and three planes determine a point unless they have a straight line in common, and three points determine a plane unless they have a straight line in common. Thus we have a peculiar interdependence underlying the concepts of the three spatial elements: the point, the straight line, and the plane. The point is descriptive of the straight line and the plane, the straight line of the point and the plane, and the plane of the straight line and the point. The point can thus be said to "lie on" a straight line or on a plane; the straight line can be said to "pass through" a point or to "lie on" a plane, and the plane can be said to "pass through" a straight line or a point.

2·1. *The Homogeneity of Space.*—By the homogeneity of space, or rather, of spatial elements is meant the possibility of identifying spatial elements by their common descriptive properties. Any two elemental points A and B have a common property, the descriptive straight line, and in this respect they are alike or identical; an elemental point A and an elemental straight line l have

a common descriptive property, the plane p, and in this respect the points of the straight line and the point A are identical. Again two elemental straight lines l' and l'' are identical in a point if they are identical in a plane, and *vice versa*. If the two straight lines l' and l'' have no point or plane in common, then the points in the straight line l' and those in the straight line l'' are identical at least in the general sense that any two points are identical in a straight line. Two elemental planes are identical in a straight line; considered as aggregates of points they are identical in the general sense also. An elemental plane and an elemental straight line are identical in a point, an elemental plane and any point O are identical in the general sense at least, if O is not in the plane; and they are identical in the straight lines through O, if O is in the plane. Thus we can identify the spatial elements X and Y in some such way, either directly by a common property, or by intermediate steps with a sequence of elements, any two successive ones having a common property.

2·2. Space is said to be heterogeneous in the sense that we have the means of distinguishing between spatial elements. Their descriptive properties render this possible.

3·1. Into the above concepts the idea of quantity does not enter. In fact they are the pre-requisites for quantity, the conditions for the possibility of co-ordinating with spatial figures the notion of quantity.

4·1. One other notion is necessary for the possibility of Descriptive Geometry in common with that of number; this is the concept of "order". Not only must there be a common trait in each type of spatial element, and a distinctive trait or traits for each element, but they must themselves be subject to *order*. Order, in the general sense, we may take to mean the disposition of SUCH elements giving to each one a particular place. The word "place" in Geometry can only refer to spatial position; so that we do not refer to *chronological* order. In a previous article, however, we indicated how TIME was implicitly connected with order, not only because physically it was required in "counting", or in passing through distinct ordered positions by a physical motion, thus giving chronological priority and subsequence, but also from a psychological necessity of time for any mental process whatever. If our elements be not subject to order nothing can be got out of them. It is an "a priori" condition, a pre-requisite to the possibility of Descriptive Geometry; and all the "a priori's" of Descriptive Geometry are also presupposed in any further Geometry. Order enables us to make also homographic or one-to-one correspondences in the spatial elements. The order in one set of elements can be imposed upon another set of elements, so that one element of one set will correspond to one and only one element of the other set mutually. The basis of this correspondence will be the spatial descriptive properties. The process of making this correspondence is called "projection".

4·2. As remarked already the introduction of order into our elements implies also the notion of spatial position. By spatial position we do not *as*

yet refer to *quantity*. The position of a spatial element is fixed when its properties distinguishing it from every other element are recognisable, and the least number of distinctive properties required is two.

4·3. Moreover, though order itself comes under the logical predicament "relation", yet the descriptive properties by which the relation is established are not themselves relations, but may be put under the logical predicament "quality". They are the basis of relations. In every relation there is the "terminus a quo", the "terminus ad quem", and the basis of the reference of the one to the other. Our terms are just mathematical concepts, and the question of "reality" can only enter into the consideration of the concrete things from which these concepts are abstracted, or to which they are applied; and these certainly are *real*. The difficult question of the reality of a relation and of the distinction of a relation from the basis of the relation, we shall, therefore, not touch upon; for the present we are building up *logically* our mathematical concepts, and the ontological question of the reality of points or straight lines or planes does not arise.

4·4. Order enables us to recognise the relative position of points or straight lines or planes. Thus we move from A to B along a straight line *l* in the sense that we mentally identify the point A with the point B by the common property *l*, also by identifying in our mental path the point A with each elemental point of the straight line in succession until we come to B. If in this motion we encounter the point C also belonging to *l*, then C is said to lie between A and B; if not, C is said to lie outside that part of the straight line *l* between A and B. Thus we have a NATURAL ORDER in the points of a range imposed by the mental process of continual identification of the points in it; and we can have an ARTIFICIAL ORDER by mentally putting these points into correspondence with an already ordered set of elements. As continuous identification of points in the straight line is possible, that is without crossing a point twice, the straight line is said to be CONTINUOUS. The number of points in the straight line is only the number of points considered. In the same way the points in a curve (it being reducible to elements—see 5.2) are also NATURALLY ORDERED in this intellectual motion. (see 5.3)

4·5. This sort of continuous order can also be attributed to the aggregate of plane straight lines in a point, the mental identification and correspondence being along any transversal, (or any second degree, closed curve enclosing the point. See n 6·2 below.); and the order that of the corresponding range so formed (or that of the natural order of the points on the curve). The same kind of continuous order can be associated with the aggregate of planes in the straight line, the correspondence being with the order of the points of any transverse range, or with that of the pencil of straight lines on any transverse plane, (or with that of any second degree closed plane curve enclosing the point). The number of points possibly existing in a straight line, or of straight lines in a point, or of planes in a straight line, or of straight lines in a plane, does not concern us.

5·1. If our Descriptive Geometry dealt only with spatial figures immediately resolvable into an integral number of elements, we have enough axioms so far to proceed with. We can build up the theory of homographic ranges and pencils and books of planes, and the theory of perspective. When we attempt to deal with figures not resolvable into a finite number of elements, we must devise a means for treating them. Thus we are forced to introduce another axiom, that of the CONTINUITY of a spatial figure within its bounds, meaning, for the present, only this: that we can reduce a "curve" into naturally ordered elemental points, any two successive points having a descriptive straight line as a property in common with the curve taken as descriptive; and also into naturally ordered elemental straight lines, any two successive ones having a descriptive point in common with the curve. The number of elements being infinite mathematically does not concern us.

5·2. A "curve" we take to mean a complex descriptive property not resolvable into a finite number of elementary descriptive points or straight lines, as well as the set of elements satisfying in the aggregate this complex descriptive property. Similarly a "curved surface" is a complex descriptive property not resolvable into a finite number of elementary descriptive planes, as well as the set of elements satisfying in the aggregate this complex descriptive property.

5·3. To deal with spatial figures involving a curve or a curved surface, we must introduce the notion of CONTINUITY, or the recognition of the existence of at least one distinct element between two distinct elements of the same type which have one common property. This notion, let it be observed, does not imply quantity, but is to be compared to the ordered continuity introduced into the number system. Between any two numbers we recognise the existence of at least one number. The number of reals between two real numbers is not enumerable, but their *natural* order is recognisable. The necessity of introducing a continuity of this kind in our concept of elements of space arises from a similar need. Without such an extension in the number system we could not introduce the irrationals into the number system. The division is not a quantitative one; it is only the recognition of the existence of elements without limit to their number in the curve, so as to enable us to reduce the curve to one type of spatial elements. We require two points of the curve giving us a descriptive property coinciding with the descriptive property of the curve, or two straight line elements of the curve giving us a descriptive point of the curve. We are not concerned with the enumerability of the elemental points or straight line elements of the curve: it will be sufficient if the curve is so reducible. In a corresponding way the necessity arises for the reducibility of curved surfaces.

5·4. Let us now summarise the axioms we make use of in the science of Descriptive Geometry.

THE POSTULATE OF THREE SPATIAL ELEMENTS. We have postulated three types of spatial elements: the plane, the straight line, and the point, into which all spatial figures are reducible.

THE AXIOM OF HOMOGENEITY AND HETEROGENEITY. In each type there is a similarity and a dissimilarity ; or we may say that each type is homogeneous, and at the same time its members are distinguishable. The basis of the identification as well as of the differentiation of the elements are the point, the straight line, and the plane, taken as descriptive. Moreover, any one of the elements can be considered as an aggregate of members of another type. Thus each type of spatial elements has been recognised as homogeneous, and a homogeneity is also found among the three different types.

THE AXIOM OF ORDER. We have acknowledged the possibility of order among our spatial elements, which are now identifiable and distinguishable. We can therefore have a homographic or one-to-one correspondence between the elements of one type, and since the types are interconnected, also between those of one type and those of another. The bases of the relationship implied by order are the descriptive point, straight line and plane. Again when analysing a spatial figure into its spatial elements, it may or may not be possible to enumerate them. Where enumeration comes in, it would correspond to mere counting, or just a correlation set up with the order of the integral number, system. There is no meaning in half a point, in half a straight line or in half a plane—at least for the present. The number of elements or of descriptive properties in any case can only be a positive whole number.

Our elements are not precisely the Euclidean elements, which are more complex concepts, and therefore the less abstract. Our concepts, however, are common to all geometrical elements, both Euclidean and non-Euclidean.

THE AXIOM OF CONTINUITY. This axiom we have introduced out of the necessity for extending our power to deal with figures not resolvable into a finite number of elements. Thus the range of Descriptive Geometry is enlarged.

6. Our apparatus of Descriptive Geometry is now completed. We are able also to get *dual* theorems, not only with the simpler class of spatial figures but also with the more complex class of curves and curved surfaces ; and this on account of the reciprocal properties of the point and the straight line, of the straight line and the plane. If all three classes be taken simultaneously, the straight line is its own reciprocal, and no duality can of course be obtained.

6.2. A curve is classified not by its quantitive characters, but by the maximum number of tangents that can be drawn to it from any point, or by the maximum number of points in which a straight line will meet it ; and thus we speak of the "class" or the "degree" of the curve. (By the *principle of duality,* we see that a curve of the second degree must also be of the second class, and è contra.) What in Metrical Geometry we distinguish as circle, parabola, ellipse, hyperbola, are all in Descriptive Geometry curves of the second degree and of the second class.

7.1. In the next article we shall show how the introduction of axioms involving quantity help us to develop our Geometry further.

ON G-FUNCTIONS IN GENERAL*

BY

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§ **1.** If $(x+1)(x+2)(x+3) \quad . \; . \; . \; (x+n) \equiv \sum_{r=0}^{n} \{G(n,\ r)x^{n-r}\}$,

then I have shown[1] that

$$G(n,\ r) = \sum_{m=1}^{r} \left\{ f_m(r).\ \binom{n+1}{2r-m+1} \right\},\quad G(n,\ 0)=1 \qquad [1°]$$

where the f's are positive integers defined recursively by

$$f_m(r) = (2r-m)\{f_m(r-1) + f_{m-1}(r-1)\}, \qquad [2°]$$

with the initial values $f_0(r)=0,\ f_1(r) = \frac{(2r)\,!}{2^r \cdot r\,!},\ f_r(r) = r\,!.$

Prof. Morgan Ward[2] has recently shown that

$$G(n,\ r) = \sum_{m=0}^{r-1} \left\{ (-1)^m \cdot H_m(r).\ \binom{n+r-m}{2r-m} \right\}, \qquad [3°]$$

where the H's are positive integers defined recursively by

$$H_m(r+1) = (2r+1-m)H_m(r) + (r+1-m)H_{m-1}(r). \qquad [4°]$$

with the initial values $H_0(r) = \frac{(2r)\,!}{2^r \cdot r\,!},\ H_{r-1}(r)=1,\ H_0(0)=1.$

§ **2.** In the present paper, I first define $G(x,\ \rho)$ for all values of x when ρ is a positive integer $\geqslant 1$. Next an expression is obtained for $G(x,\ \rho)$ when x is a negative integer $\leqslant -1$, ρ being a positive integer $\geqslant 1$.

Lastly $G(x,\ \rho)$ is defined for all values of ρ, and an interesting theorem is proved. The results of this paper are of value in so far as they have enabled me to obtain the exact form of Ward's Numbers.

1. Hansraj Gupta, "Sums of Products of first n natural numbers taken r at a time."—*Journal of the Indian Mathematical Society.* Part II, Vol. XIX, pp. 1-6. (1931).

2. Morgan Ward, "Sterling's Numbers and Polynomials"—*American Journal of Mathematics.* Vol. 56, (1934), pp. 87—95.

3. In my paper on Professor Ward's Numbers. (In Press.)

I have found[3] that

$$(m-1)!\ \mathrm{H}_{r-m}(r)=\sum_{k=0}^{m-1}\{(-1)\theta_k(r,m)\cdot(m-k)^{r+m-k-1}\},$$

where

$$\theta_k(r,\ m)=\binom{m-1}{k}\left\{(m-k)^k+\binom{k}{1}(m-k)^{k-1}(r+m)^{(1)}\right.$$
$$\left.+\binom{k}{2}(m-k)^{k-2}(r+m)^{(2)}+\ .\ .\ .\ +\binom{k}{k}(r+m)^{(k)}\right\},$$

$j^{(i)}$ denoting as usual $j(j-1)(j-2)\ .\ .\ .\ (j-i+1)$.

In what follows we use k, m, n, r for positive integers, and x and ρ for all numbers. The fundamental relation satisfied by the G-functions is

$$\mathrm{G}(x+1,\ \rho)=\mathrm{G}(x,\ \rho)+(x+1)\ \mathrm{G}(x,\ \rho-1). \qquad \text{(F)}$$

§ **3.** For all values of x, we take

$$\mathrm{G}(x,\ r)\equiv\sum_{m=1}^{r}f_m(r)\cdot\binom{x+1}{2r-m+1},\qquad r\geqslant 1 \qquad [5^\circ]$$

where $f_m(r)=(2r-m)\{f_m(r-1)+f_{m-1}(r-1)\}$, $f_0(r)=0$, $f_1(r)=\dfrac{(2r)!}{2^r\cdot r!}$.

Evidently $\mathrm{G}(x,\ r)$ is a rational algebraic function of degree $2r$ satisfying the fundamental relation (F).

Since $\mathrm{G}(n,\ r)=0$ when $0\leqslant n<r$, $r\geqslant 1$, the expression for $\mathrm{G}(x,\ r)$ must have x, $x-1$, $x-2$, ..., $x-r+1$, as some of its algebraic factors. In particular $\mathrm{G}(x+1,\ r)$ must have $(x+1)$ as a factor. From the fundamental relation, it follows that $\mathrm{G}(x,\ r)$ has $(x+1)$ also as an algebraic factor.

Hence

$$\mathrm{G}(x,\ r)=\frac{1}{(2r)!}\,(x+1)\,x\,(x-1)\ldots(x-r+1)\{a_1\,x^{r-1}+a_2\,x^{r-2}+\cdot\cdot\cdot+a_r\},$$

where the a's are positive or negative integers and $r\geqslant 1$. [6°]

Thus $\mathrm{G}(-1,\ r)=0,\qquad r\geqslant 1.$ [7°]

§ **4.** To find an expression for $\mathrm{G}(-n-1,\ r)$, $\qquad n\geqslant 1,\ r\geqslant 0.$

In the fundamental relation (F) put -2 for x, then $\mathrm{G}(-1,\ r)=\mathrm{G}(-2,\ r)-\mathrm{G}(-2,\ r-1)$; since $\mathrm{G}(-1,\ r)=0,\ r\geqslant 1$; we get $\mathrm{G}(-2,\ r)=\mathrm{G}(-2,\ r-1)=\mathrm{G}(-2,\ r-2)=\ldots\ldots$

$$=\mathrm{G}(-2,\ 1)\quad=\mathrm{G}(-2,\ 0)\quad=1. \qquad [8^\circ\cdot 1]$$

Putting -3 for x in (F), we obtain

$$G(-2, r) = G(-3, r) - 2G(-3, r-1),$$
$$G(-2, r-1) = G(-3, r-1) - 2G(-3, r-2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

and $$G(-2, 1) = G(-3, 1) - 2G(-3, 0),$$

where $G(-3, 0) = 1$.

Multiplying these equations in order by $1, 2, 2^2, \dots, 2^{r-1}$, and adding the results, we have

$$G(-3, r) - 2^r G(-3, 0) = 1 + 2 + 2^2 + \dots + 2^{r-1}, \quad \text{Since } G(-2, r) = 1$$

Hence $G(-3, r) = 2^{r+1} - 1$. [8°·2]

Again putting -4 for x in (F), we get

$$G(-3, r) = G(-4, r) - 3G(-4, r-1),$$
$$G(-3, r-1) = G(-4, r-1) - 3G(-4, r-2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

and $$G(-3, 1) = G(-4, 1) - 3G(-4, 0), \quad \text{where } G(-4, 0) = 1.$$

Multiplying these equations in order by $1, 3, 3^2, \dots, 3^{r-1}$, and adding, we have

$$G(-4, r) = \sum_{k=1}^{r+1} \{3^{k-1} \cdot G(-3, r-k+1)\},$$
$$= \sum_{k=1}^{r+1} \{3^{k-1} (2^{r-k+2} - 1)\},$$
$$= \frac{1}{2!} 3^{r+2} - \frac{1}{1!\,1!} 2^{r+2} + \frac{1}{2!} 1^{r+2}. \qquad [8°·3]$$

Proceeding thus we can easily prove that

$$G(-n-1, r) = \frac{1}{(n-1)!} n^{r+n-1} - \frac{1}{(n-2)!\,1!} (n-1)^{r+n-1} + \frac{1}{(n-3)!\,2!} (n-2)^{r+n-1} - \dots + (-1)^{n-1} \cdot \frac{1}{(n-1)!} \cdot 1^{r+n-1};$$

so that

$$(n-1)!\; G(-n-1, r) = \sum_{k=0}^{n-1} \{(-1)^k \cdot \binom{n-1}{k} \cdot (n-k)^{r+n-1}\}. \qquad [8°]$$

Since the result holds when $r=0$, and $G(-n-1, 0)=1$; $n \geqslant 1$, we have

$$(n-1)!=\binom{n-1}{0} \cdot n^{n-1}-\binom{n-1}{1} \cdot (n-1)^{n-1}+\binom{n-1}{2} \cdot (n-2)^{n-1}$$
$$- \dots +(-1)^{n-1} \cdot \binom{n-1}{n-1} \cdot 1^{n-1}. \qquad [8^{\circ}\cdot 4]$$

§ **5.** We now define $G(-n-2, \rho)$, $n \geqslant 1$, for all values of ρ by the relation

$$n!\, G(-n-2, \rho) = \sum_{k=0}^{n} \left\{ (-1)^k \cdot \binom{n}{k} \cdot (n-k+1)^{n+\rho} \right\} \qquad [8^{\circ}\cdot 5]$$

and consider the values of $G(-n-2, -r)$ for values of n and $r \geqslant 1$.

We notice that
$$G(-3, -r)=0 \quad \text{when } r=1;$$
$$G(-4, -r)=0 \quad \text{when } r=1, 2;$$
$$G(-5, -r)=0 \quad \text{when } r=1, 2, 3.$$

In general $G(-n-2, -r)=0$, when $r=1, 2, 3, \dots, n$. $\qquad [9^{\circ}]$

To prove this, we suppose that
$$G(-m-1, -r)=0, \text{ when } r=1, 2, 3, \dots, m-1; \qquad m \leqslant n.$$

Then the fundamental relation can be written in the form
$$(m+1)\, G(-m-2, -r)=G(-m-2, -r+1)-G(-m-1, -r+1).$$

In particular
$$(m+1)\, G(-m-2, -1)=G(-m-2, 0)-G(-m-1, 0)=0, \qquad m \geqslant 1.$$

so that $\quad G(-m-2, -1)=0.$

Now $(m+1)\, G(-m-2, -2)=G(-m-2, -1)-G(-m-1, -1)=0, \quad m \geqslant 2.$

therefore $\quad G(-m-2, -2)=0.$

Similarly $\quad G(-m-2, -3)=0, \qquad m \geqslant 3.$

....

and $\quad G(-m-2, -m+1)=0.$

Finally
$$(m+1)\, G(-m-2, -m)=G(-m-2, -m+1)-G(-m-1, -m+1)=0$$
so that $\quad G(-m-2, -m)=0. \qquad m \geqslant 1.$

The theorem now follows at once by induction.

Moreover

$$(m+1)\,G(-m-2,\ -m-1)=G(-m-2,\ -m)-G(-m-1,\ -m),$$

or $$G(-m-2,\ -m-1)=-\frac{1}{m+1}\cdot G(-m-1,\ -m),\qquad m\geqslant 2. \qquad [10°]$$

Hence

$$G(-n-1,\ -n)=\frac{(-1)^{n+1}}{n\,!},\quad \text{because } G(-3,\ -2)=-\tfrac{1}{2}. \qquad [11°]$$

Also from [8°], we get

$$n\,!\ G(-n-1,\ r)=\sum_{k=0}^{n-1}\left\{\binom{n}{k}\cdot(-1)^k\cdot(n-k)^{r+n}\right\}, \qquad [8°]$$

so that $$n\,!\ G(-n-1,\ \rho)=\sum_{k=0}^{n-1}\left\{\binom{n}{k}\cdot(-1)^k\cdot(n-k)^{\rho+n}\right\}\cdot \qquad [8°\cdot 5']$$

Hence

$$n\,!\ G(-n-1,\ -n)=\sum_{k=0}^{n-1}\left\{(-1)^k\cdot\binom{n}{k}\right\}=(-1)^{n+1},$$

or $$G(-n-1,\ -n)=\frac{(-1)^{n+1}}{n\,!},\qquad n\geqslant 2, \text{ as before.}$$

§ 6. The a's in [6°] can be determined easily with the help of the results of § 4.

Thus $G(x,3)=\frac{1}{6\,!}(x+1)\,x\,(x-1)\,(x-2)\,\{a_1x^2+a_2x+a_3\}$.

Put $-2,\ -3,\ -4$ for x in succession.

Then $$1=\frac{1}{30}\{4a_1-2a_2+a_3\},$$

$$2^4-1=\frac{1}{6}\{9a_1-3a_2+a_3\},$$

and $$\frac{1}{2\,!}\,3^5-\frac{1}{1!\ 1!}\,2^5+\frac{1}{2\,!}\,1^5=\frac{1}{2}\,\{16a_1-4a_2+a_3\}.$$

Hence $a_1=15,\ a_2=15,\ a_3=0$, so that

$$G(x,\ 3)=\frac{15}{6\,!}\,(x+1)^2\ x^2(x-1)\,(x-2).$$

§ **7.** From the following scheme it will appear that we may in view of the fundamental relation, take

$$\left.\begin{array}{lll} G(-2, \pm r) = 1, & & r \geqslant 0. \\ G(-1, \pm r) = 0, & & r \geqslant 1. \\ G(n, 0) & = 1, & n \geqslant 0. \\ G(-n, 0) & = 1, & n \geqslant 2. \\ G(n, -r) & = 0, & r \geqslant 1. \end{array}\right\} \qquad [12^\circ]$$

$G(-1, 0)$ remains ambiguous and may be -1, 0, or 1 [*cf.* 11°, 7°, 1°]

§ **8.** Values of $G(x, \rho)$ for integral values of x, ρ; $|x| \leqslant 5$, $|\rho| \leqslant 4$.

1701	301	31	1	0	0	0	0	0	24	274
350	90	15	1	0	0	0	0	6	50	225
65	25	7	1	0	0	0	2	11	35	85
10	6	3	1	0	0	1	3	6	10	15
1	1	1	1	?	1	1	1	1	1	1
0	0	0	1	0	0	0	0	0	0	0
0	0	$-\frac{1}{2}$	1	0	0	0	0	0	0	0
0	$\frac{1}{6}$	$-\frac{3}{4}$	1	0	0	0	0	0	0	0
$-\frac{1}{24}$	$\frac{11}{36}$	$-\frac{7}{8}$	1	0	0	0	0	0	0	0

GLEANINGS

37. Had the mathematical concept of limit been known in the days of Plato, the great thinkers shining Absolutes—absolute justice, absolute beauty, absolute truth, absolute good—whose "perception by pure intelligence" brings us, said he, "to the end of the intellectual world," would not have appeared to him as "ends" or final terms of any sequences or progressions within the intellectual world, but rather as limits or ideals above and beyond it, as downward looking aspects of an over-world.

C. J. KEYSER, Essay on, *The bearings of Mathematics.*

FOCAL LINES OF A CONE TOUCHING FOUR GIVEN CONCURRENT PLANES

BY

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1. The main inspiration of this paper is Mr. V. Ramaswami Aiyar's budget of theorems on "Circular cylinders circumscribing a Tetrahedron" * with the remark that he was not able to reduce his fundamental theorem by purely geometrical considerations. In this paper we propose to prove the following theorem by purely synthetic methods.

THEOREM: *The axes of circular cylinders circumscribing a tetrahedron are parallel to the focal lines of a cone touching four concurrent planes parallel to the faces of the tetrahedron.*

This is another variant of Mr. Aiyar's fundamental theorem.

In this connection, it may be pointed out that the cubic cone which is the locus of these focal lines is the three-dimensional analogue of the well-known cubic which is the locus of the foci of conics touching four given lines †. If we remember that any two opposite sides of a quadrilateral circumscribing a conic subtend equal or supplementary angles at a focus, the plane cubic aforesaid may also be considered as the locus of the common vertex of two triangles whose bases are given and vertical angles equal or supplementary; Salmon has shown that the latter locus is a curve of the third degree passing through the circular points at infinity ‡. This same problem has also been elaborately dealt with by Mr. M. T. Naraniengar in *J. I. M. S.*, Vol. III, pp. 132-140, where great pains are taken to show that the equation (obtained by him) is really a cubic.

2. Let ABCD be the given tetrahedron and Π any plane which is cut by the faces ABC, ACD, ADB, and the plane through A parallel to the face BCD in the lines QR, RS, SP, PQ respectively. We shall now consider the parallel projections A′, B′, C′, D′ of A, B, C, D respectively on the Π-plane.

* *Mathematics Student* pp. 89-91, Vol. II, No. 3.

† Salmon: *Conic Sections* p. 275, where the equation to the locus is given in the form $a/\alpha + b/\beta + c/r + d/\delta = 0$, exactly resembling Mr. Aiyar's equation (3·2) on p. 9?, M. S. Vol. II.

‡ *Vide* Salmon: *Higher Plane Curves*, p. 142.

If L, M, N be the points where CD, DB, BC meet the Π-plane, they are collinear and lie respectively on RS, SP, QR; and MN is parallel to PQ. If PQ, RS meet in T, then AT is parallel to CD and since parallel lines project into parallel lines, C′ D′ is parallel to A′T and passes through L. Similarly D′ B′ passes through M and is parallel to A′P; and B′ C′ passes through N and is parallel to A′Q.

The configuration thus brought down from three to two dimensions is reduced to this:

PQRS is a plane quadrilateral and A′ is any point in its plane, PQ, RS meet in T and a straight line parallel to PQ meets RS, SP, QR in L, M, N respectively. Lines C′ D′, D′ B′, B′ C′ are drawn through L, M, N parallel respectively to A′T, A′P, A′Q so as to form the triangle B′ C′ D′.

Now the triangles TA′Q, LC′N are similar and similarly situated, so that A′C′ passes through R; similarly, A′D′ passes through S. Hence ∠ C′A′D′ = ∠ SA′R.

Further, if A′, B′, C′, D′ be concyclic, C′ D′ will subtend equal or supplementary angles at A′ and B′, and in this case, it follows that PQ, SR subtend equal or supplementary angles at A′ (angles PA′Q and C′ B′ D′ being obviously equal).

Conversely, it is easily seen that if PQ, SR subtend equal or supplementary angles at A′, then A′, B′, C′, D′ are concyclic.

This last result is the key to Mr. Aiyar's theorem.

If A, B, C, D are projected orthogonally on to the Π-plane and the projections denoted by A′, B′, C′, D′ be concyclic, then PQ, RS will subtend equal or supplementary angles at A′ and therefore with A′ as a focus, a conic Γ can be drawn to touch the sides of the quadrilateral PQRS.

Now, the cone with vertex A passing through the conic Γ touches the planes APQ, AQR, ARS, ASP and has AA′ as a focal line; for the section of this cone by the plane perpendicular to AA′ has A′ as a focus *; AA′ is evidently parallel to the axis of a circular cylinder circumscribing the tetrahedron ABCD.

Hence the theorem enunciated in § 1.

* Baker, *Principles of Geometry* p. 109, Vol. III.

3. From the fundamental property of a focal line of a cone viz. a plane at right angles to the focal line meets the cone in a conic of which one focus is the intersection of the line with the plane, we rapidly derive a number of interesting results.

(i) Let S be any point on a focal line of a cone with vertex A; and let the perpendicular from S on a tangent plane to the cone meet it in Q. If AQ meet the Π plane perpendicular to AS in P, $\angle$ ASP is a right angle and $AQ.AP = AS^2$. Hence the locus of Q is the inverse of the locus of P. But since the plane ASP is perpendicular to both the tangent plane and the Π plane, SP is perpendicular to their common line of intersection, which is evidently a tangent to the conic Γ in which the plane Π cuts the cone. Thus P lies on the auxiliary circle of Γ and hence the locus of Q is also a circle, being the inverse w. r. t. A of this auxiliary circle. *

(ii) Let the normal at A to the tangent plane meet the Π plane in P′. Then since $SP, SP' = AS^2$, the locus of P′ is also a circle. But AP′ is a generator of the reciprocal of the given cone and the locus of P′ is therefore a circular section of this reciprocal cone. Hence the well-known theorem: The focal lines of a cone are perpendicular to the cyclic planes of the reciprocal cone.

(iii) If AS′ is a focal line associated with AS, the product of the perpendiculars from S, S′ on any tangent plane to the given cone is constant.

For, let AP′ be the normal to any tangent plane and ST be drawn perpendicular to AP′ to meet it in T. Then $AS^2/AP' = AT =$ the perpendicular from S on the tangent plane. Similarly, the perpendicular from S′ on the same tangent plane $= AS'^2/AP''$ P″ being the point where AP′ meets the plane through S′ perpendicular to AS′. Since P′, P″ are on two complementary circular sections which lie on a sphere †, AP′. AP″ is constant. Hence the product of the perpendiculars from the foci S, S′ of a cone on any tangent plane is also constant.

(iv) The above property is exactly analogous to that of the foci of a conic. The problem of finding the locus of the focal lines of a cone touching four given concurrent tangent planes is algebraically equivalent to finding the locus of the foci of cones touching four given lines.

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* This result was pointed out by Mr. Aiyar. (*Vide* p. 91. M. S. Vol. II.)

† Baker, *Principles of Geometry* Vol. III. p. 83. Ex. 2.

If p_1, p_2, p_3, p_4 be the perpendiculars from any point say S, on four given concurrent planes, it is easy to see that there is an identical relation connecting them, in the form

$$Ap_1 + Bp_2 + Cp_3 + Dp_4 \equiv 0$$

If with S, another point S′ can be associated such that $p_1 p'_1 = p_2 p'_2 = p_3 p'_3 = p_4 p'_4$ where the primed $p's$ are the perpendiculars, from S′ on the same planes, then

$$A/p'_1 + B/p'_2 + C/p'_3 + D/p'_4 = 0$$

Similarly also $A/p_1 + B/p_2 + C/p_3 + D/p_4 = 0.$

Taking the equations of the given planes in the form

$U_r \equiv x \cos \alpha_r + y \cos \beta_r + z \cos \gamma_r = 0$ ($r = 1, 2, 3, 4$ and $\sum \cos^2 \alpha_r = 1$),

we see that S, S′ both lie on the cone

$$\frac{A}{U_1} + \frac{B}{U_2} + \frac{C}{U_3} + \frac{D}{U_4} = 0 \qquad \ldots\ldots\ldots(1)$$

This equation may also be regarded as representing the locus of the focal lines of a cone touching the four planes $U_r = 0$ ($r = 1, 2, 3, 4$).

4. The cubic cone (1) intersects any plane,

$$lx + my + nz = p \qquad \ldots(2)$$

in a cubic, which is projectively equivalent to the cubic in which the cone cut the plane at infinity. From the properties of focal lines the points in which they cut the plane at infinity are the intersections of the common tangents of the circle at infinity and the conic in which the given cone meets the plane at infinity. Hence, the corresponding property is also true on the plane (2), so that the cubic in question is the locus of the points of intersection of the common tangents of a fixed conic on (2) and a variable conic belonging to the four-line system of conics in which the cones touching the given planes cut the plane (2). Since pairs of tangents from a given point to a four-line system of conics are in involution, the cubic locus may be regarded as the locus of points such that the pairs of tangents from them to three given conics are in involution.*

* *Vide* Salmon's Conic Sections p. 364.

DRIKKARMA CORRECTION IN HINDU ASTRONOMY

BY

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Two planets are said to be in conjunction in Hindu Astronomy not when they have the same longitude but when they are upon the same secondary to the prime Vertical. This naturally introduced a correction called दृक्कर्म *Drikkarma* in the longitudes of the planets. In this paper, we propose to discuss its nature and the way in which it was calculated by the Hindus.

Let S be the position of a planet in its orbit विमण्डल *Vimandala*, K the pole of the Ecliptic कदम्ब *Kadamba*, P the pole of the Equator ध्रुव *Dhruva* and n the pole of the prime vertical or north point. Since the angles $\angle$ KSP, $\angle$ PSN are of use in obtaining the value of the above Drikkarma correction, it is convenient to calculate them first. From a well-known formula in Spherical Trigonometry, we have from the triangle KSP,

$$\frac{\sin \text{ KSP}}{\sin \omega} = \frac{\sin(90-\lambda)}{\sin(90-d)}$$

where λ, d and ω are respectively the longitude, declination and the obliquity of the Ecliptic.

$$\therefore \quad \sin \text{KSP} = \sin \omega \cos \lambda / \cos d.$$

This angle is called in Hindu Astronomy आयनवलनं *Ayanavalana* i.e., deflection due to obliquity of the Ecliptic.

Again from the triangle PSn,

$$\frac{\sin \text{PS}n}{\sin \phi} = \frac{\sin \text{P}n\text{S}}{\cos d} = \frac{\sin nPS}{\sin nS}$$

where ϕ is the latitude of the place.

$$\therefore \quad \sin \text{PS}n = \frac{\sin \phi \sin \text{P}nS}{\cos d} = \frac{\sin \phi \sin h}{\cos \mu}$$

Just as $\cos d$ where d is the declination of the planet represents the radius of the diurnal circle, and hence is called द्युज्या *Dhujya*, similarly $\cos \mu$ where μ represents the planet's perpendicular distance from the Prime Vertical represents the radius of the उपवृत्त *upavritta*, a circle through the body parallel to the prime vertical or सममण्डल *samamandala*.

The angle PSn above, is called the **आक्षवलनं** *Akshavalana* or the deflection due to the latitude. In particular, sin PSn reduces to sin ϕ/cos d when the planet is on the horizon.

If the point S represents the **बिम्बकेन्द्र** *bimbakendra* or the centre of the planet's disc, these valanas or deflections are termed the **बिम्बीयायनवलन**, and the **बिम्बीयाक्षवलनम्**. The corresponding deflections for another position of the planet, namely the point on the Ecliptic, which represents the planet's longitude, can be easily got, the change in this case being that we have to take the **अस्फुटक्रान्ति** *asputakranti* of the planet instead of its **स्फुटक्रान्ति** * *sputakranti*. These deflections are termed the **स्थानीय** or **चिह्नीय** *valanas*.

The same formulae as obtained above for these two corrections, are given by Bhaskara by the following slokas in *Goladhyaya*, *Valana Vasana*.

(i) ......खेटकोटिक्रमज्यका, जिनज्याघ्नी, द्युजीवाप्ता आयनदिग्वलनं भवेत् ।

(ii) अग्रानृतलयोर्योगः समदिक्त्वेऽन्यथान्तरम् ।
तत्त्रिज्यावर्गविश्लेषपदभक्ताक्षशिञ्जिनी ।
नतासुदोर्ज्यया क्षुण्णा वलनं पलजं स्फुटम् ॥

i.e. (1) The cosine of the planet's longitude multiplied by the sine of 24° (i.e. the obliquity of the Ecliptic as taken by the Hindus) and divided by the cosine of the declination becomes the *ayanavalana*.

(2) The sum of the *Agra* and *Sankuthala* (which is equivalent to the *bahu* or the distance of the planet from the Prime Vertical) or the difference, as the case may be, is obtained; then sin ϕ divided by $\sqrt{R^2 - \text{bahu}^2}$ and multiplied by the sine of the hour angle gives the sine of the Akshavalana. Here, of course $\sqrt{R^2 - \text{bahu}^2}$ stands for the cos μ written above or the radius of the *upavritta* as it is called.

In practice, since it is difficult to obtain the value of the *bahu*, the Hindu Astronomers gave various other methods as a short cut, which are therefore less rigorous. But it will be seen as from Bhaskara, that it is not due to their incompetency in Mathematics, as some have hastily concluded.

* In Hindu Astronomy, the declination of any celestial body is termed *sputakranti*, while the arc intercepted between the Equator and the Ecliptic on its declination circle is termed *asputakranti* or simply *kranti*. Many writers on Hindu Astronomy seem to have overlooked this distinction.

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As an alternative to the above-cited sloka for Akshavalana, he gave the following one (*Gaintadhyaya Chandragrahanadhikara*), which Bhaskara gave obviously as an approximate value, not that he did not know how to calculate the correct one.

"खाङ्काहतं स्वद्युदळेन भक्तं स्पर्शादिकालोत्थनतं लवाः स्युः ।
तेषां क्रमज्या पलशिञ्जिनीघ्नी भक्ता द्युमौर्व्या यदवाप्तचापम् ॥
प्रजायते प्रागपरे नते क्रमात् उदग्यमाशं वलनं पलोद्भवम् ।

which implies that the diurnal circle is approximately taken to be parallel to the prime vertical (**सममण्डल**). Strictly speaking, the angle PnS which is required in the formula sine PSn = Sin ϕ sin PnS/cos$d$ given before, ought to be calculated only by the following formulae obtained from the spherical triangle PnS, namely

$$\text{(i)} \quad \cos x = \frac{\sin d - \sin \mu \cos \phi}{\cos \mu \sin \phi}$$

$$\text{(ii)} \quad \cot x = \frac{1}{\sin h}\left\{ \sin \phi \tan d - \cos h \cos \phi \right\}.$$

But since this calculation necessitates a good deal of labour, as Bhaskara remarks in *Valana Vasana*, *Goladhyaya*, the above procedure given in the Sloka '**खाङ्काहतं**, etc.' is adopted as a short cut.

In Suryasiddhanta, the formulae given for these valanas are as follows.

(i) नतज्याक्षज्ययाभ्यस्ता, त्रिज्याप्ता तस्य कार्मुकम्,
वलनांशाः सौम्ययाम्याः पूर्वापरकपालयोः ।
राशित्रययुतात् ग्राह्यात् क्रान्त्यंशैः दिक्समैः युताः,
भेदेंतरात् ज्या वलना सप्तत्यङ्गुलभाजिता ॥

i.e. $\sin h \sin \phi / \text{R} = \sin$ (Akshavalana)

and sine (Ayanavalana) = sine of the declination of a point which is ahead of the planet in the Ecliptic by 90°, i.e. Substituting $(90 + \lambda)$ for λ in the well-known formula $\sin d = \sin \omega \sin \lambda$ giving

$$\sin d = \sin \omega \cos \lambda.$$

In both of these formulae, it is seen that cos d is taken to be R as an approximation which amounts to neglecting the declination of the planet. Having thus obtained the two angles of the valana, let us see now how the *Drikkarma* correction is calculated by the Hindus.

In the Suryasiddhanta VIIth canto, the formulae given for Aksha and Ayana Drikkarmas are as follows.

(i) विषुवच्छाययाभ्यस्तात् विक्षेपात् द्वादशोद्धृतात्
फलं स्वनतनाडीघ्नं स्वदिनार्धविभाजितम् ।
लब्धं प्राच्यामृणं सौम्यात् विक्षेपात् पश्चिमे धनं
दक्षिणे प्राक्कपाले स्वं पश्चिमे तु तथा क्षयः ॥

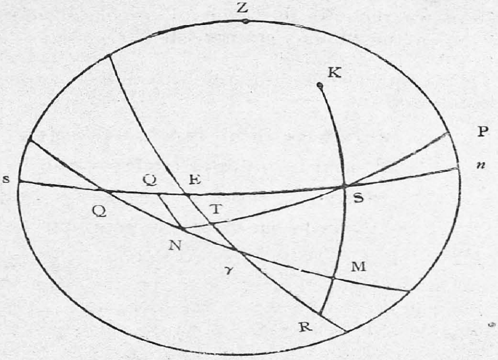
(ii) सत्रिभग्रहजक्रान्तिभागघ्नाः क्षेपलिप्तिकाः ।
विकलाः स्वमृणं क्रान्तिक्षेपयोः भिन्नतुल्ययोः ॥

i.e. (i) $\dfrac{\text{celestial latitude} \times \tan\phi \sin h}{\sin H}$ = Aksha Drikkarma,

where H is the hour angle of the planet on the horizon.

(ii) Ayana Drikkarma correction in seconds of arc = number of degrees in ($\sin\omega \cos\lambda$) multiplied by number of minutes of arc in latitude. They may be proved as follows.

Let S be the position of the planet on the horizon, let P, K, be respectively the poles of the Equator and the Ecliptic; let M, N, Q be respectively the points of intersection of the Ecliptic with the latitude circle, declination circle and the horizon (horizon is here the particular case of what is generally called the circle of position समप्रोतवृत्त i. e., a secondary to the prime vertical through the planet). Let T, R, be the points of intersection of the Equator with the planet's declination and latitude circles. Then M represents the planets' position on the Ecliptic; N the *Ayanagraha* i.e. the planet corrected for *Ayanadrikkarma*; Q the planet corrected for *Aksha-Drikkarma*; so that



Arc MN = Ayana Drikkarma Correction.

Arc NQ = Aksha Drikkarma Correction.

The formula given for the Aksha Drikkarma above takes the arc NQ equal to the arc NQ′ and as a second approximation takes SN equal to SM. Hence from the right-angled triangle SNQ′ wherein SN = d (say) $\angle SQ'N = 90 - \phi$

$$\sin Q'N = \tan SN \tan \phi.$$

Again as a third approximation tan SN is taken to be SN itself, which implies that the latitude of the planet is very small; similarly sin Q′N is taken to be Q′N itself. Hence, finally

$$Q'N = SN \times \tan \phi.$$

Thus Q′N, which is taken to be the Aksha Drikkarma correction as a first approximation, has the value as given in the sloka, SN standing for SM as an approximation.

Again, since this is the correction when the planet is on the horizon, and since the amount of correction, when the planet is on the meridian is nil, N and Q then coinciding, the amount of correction when the planet is anywhere else in its diurnal path, is taken to vary as the sine of the hour angle i.e. if when the hour angle is equal to H the hour angle on the horizon the correction is as above, then what is the correction when the hour angle assumes a value h? The answer is

$$\frac{\sin h}{\sin H} \times \tan \phi \times \text{Vikshepa.}$$

Here the ratio taken in the above Sloka is in time sin H corresponding to what the Hindus call 'half day of the planet'. Obviously the approximations made are far from rigorous but a more approximate value could have been obtained had they made use of the formula sine Akshadrikkarma = sine SN × tan ϕ instead of taking sine SN equal to SN. As regards the value of the correction for a different position of the planet, the proportion applied implies that the diurnal path is taken to be parallel to the prime vertical as in Akshavalana, and is only an approximation. Since Bhaskara gave rigorous formulae for these corrections, it must not be generalized as done by Burgess in the translation of this Sloka, in Suryasiddhanta, that the Hindus were not competent to obtain a more rigorous estimate of the correction. We can only say that they were satisfied with that amount of accuracy, even though they were competent to obtain a more rigorous value, just to save labour.

As regards the Ayanadrikkarma, the formula is correct, provided we understand $\frac{\sin \omega \cos \lambda}{R}$ is taken as an approximation to $\frac{\sin \omega \cos \lambda}{\cos d}$ in obtaining Ayanavalanam; for taking the triangle SNM, we see sin SM = tan NM/tan NSM or since ∠ NSM is small, and similarly NM,

$$NM = \sin \angle NSM \times \sin SM.$$

But since ∠ NSM is seen to be $\sin \omega \cos \lambda / \cos d$, or approximately $(\sin \omega \cos \lambda)/R$, the formula finally assumes the form given in Suryasiddhanta. The injunction to take the declination in degrees and latitude in minutes is due to transforming the formula into seconds of arc, which necessitates multiplication by 3438×60 i.e. $R \times 60$, which the Hindus did. The idea that the Hindus multiplied an expression by 60 R to transform radian measure into seconds of arc, is made clear in my paper on Hindu Trigonometry, published in the *Mathematics Student*, Vol. 1, No. 2, 1934.

NOTES AND DISCUSSIONS

A Biquadrate as the sum of four Biquadrates.*

1. In confirmation of Euler's conjecture that a fourth power can be the sum of four fourth powers R. Norrie found in 1911 by a series of special assumptions the single result†

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4.$$

I shall give here a simple method of deriving this interesting result.

2. We have $x^4 - y^4 = (x+y)(x-y)(x^2+y^2)$ I

Taking $x + y = a^4$ and $x - y = b^4$ we get from I

$$\left(\frac{a^4+b^4}{2}\right)^4 = \left(\frac{a^4-b^4}{2}\right)^4 + a^4b^4\left(\frac{a^8+b^8}{2}\right)$$

If $\frac{a^8+b^8}{2}$ can be expressed as the sum of three fourth powers, then it is possible to find examples of a fourth power as the sum of four fourth powers.

* I wish to express my sincere thanks to Dr. S. Chowla for the kind interest he has taken in this work.

† Dickson; *History of the Theory of Numbers*, Vol. 2, p. 652. Norrie's paper has been inaccessible to me.

This is possible in the particular case when $a=5$ and $b=3$.

For, $$\frac{5^8+3^8}{2}=21^4+8^4+2^4$$

Substituting these values in I, we have

$$\left(\frac{5^4+3^4}{2}\right)^4=\left(\frac{5^4-3^4}{2}\right)^4+5^4\,3^4\,(21^4+8^4+2^4)$$

or $$353^4=272^4+315^4+120^4+30^4$$

Benares Hindu University, BENARES. S. SASTRY.

A Magic Hypercube

1,	41,	81,	50,	63,	10,	72,	19,	32,
68,	27,	28,	9,	37,	77,	46,	59,	18,
54,	55,	14,	64,	23,	36,	5,	45,	73,
44,	75,	4,	57,	13,	53,	22,	35,	66,
21,	31,	71,	43,	80,	3,	62,	12,	49,
58,	17,	48,	26,	30,	67,	39,	76,	8,
78,	7,	38,	16,	47,	60,	29,	69,	25,
34,	65,	24,	74,	6,	43,	15,	52,	56,
11,	51,	61,	33,	70,	20,	79,	2,	42,

In the above hypercube, as I call it, all the first $3^4=81$ integers are used without repetition. The properties are

(1) The sum of the three figures in any row or column of any square of the hypercube is 123.

(2) The sum of any set of the three corresponding figures taken from three squares in any row or column of the hypercube is also 123.

Kottayam. P. A. EAPPEN

An Integral Test for the Convergence of a Series of Positive Terms *

In a paper contributed to the *American Mathematical Monthly*, Volume 38 (1931), Professor Raymond W. Brink has established the following 'integral test' for a series of positive terms :—

Given the series $u_1 + u_2 + u_3 + \cdots$ $(u_n > 0,\ n \geqslant m)$, let $r(x)$ be a function with a continuous derivative $r'(x)$, $x \geqslant m$, such that $\int^{\infty} |r'(x)|dx$ converges and $r(n) = u_{n+1}/u_n$. Then the convergence of the integral $\int^{\infty} e^{\int_m^x [r(x)-1]dx}\, dx$ is sufficient for the convergence of the series; and the divergence of the integral $\int^{\infty} e^{\int_m^x \left[1-\frac{1}{r(x)}\right]dx}\, dx$ is sufficient for the divergence of the series.

Professor Brink has also shown that this test can be derived from another test published by him in the *Transactions of the American Mathematical Society*, Volume 19 (1918):—

$\sum u_n$ and $r(x)$ satisfying the conditions already specified, the convergence (or divergence) of $\int^{\infty} e^{\int_m^n \log r(x)\, dx}\, dx$ implies the convergence (or divergence) of the series.†

It is the object of this note to show that either test is deducible from a theorem which may be formulated as follows: The convergence [or divergence] of the integral $I = \int^{\infty} e^{\int_m^x \phi(x)dx}\, dx$ is sufficient for the convergence (or divergence) of $\sum^{\infty} u_n$ $(u_n > o,\ n \geqslant m)$ under the following circumstances :—

(1) $\phi(x)$ is uniformly bounded and integrable in the interval (m, m') where $m' > m$ is arbitrary.

(2) $\phi(x) \geqslant \log r(n) + v_n$ [or $\phi(x) \leqslant \log r(n) + v_n$], for $n \leqslant x \leqslant n+1$, where v_n is the general term of an absolutely convergent series. (The alternative in square brackets gives a test for the divergence of $\sum u_n$).

* My thanks are due to Mr. G. A. Srinivasan of whose help I have freely availed myself in the preparation of this note.

† The significance of these tests may be said to lie in the fact that they include not only the sequence of de Morgan's and Bertrand's tests (*vide* the papers referred to above) but also the sequence of logarithmic tests :—

If $\log \frac{1}{r(n)} = \frac{1}{n} + \frac{1}{nl_1 n} + \cdots + \frac{\sigma(n)}{nl_1 n \cdots l_k n}$ $(l_1 n = \log n,\ l_2 n = \log \log n$ etc.),

the series $\sum^{\infty} u_n$ is convergent when $\lim\limits_{n\to\infty} \sigma(n) > 1$ and divergent when $\lim \sigma(n) < 1$.

Proof. From (2), by integration

$$\int_m^n \phi(x)dx \geqslant \log r(m)\cdot r(m+1) \cdots r(n-1) + v_m + v_{m+1} + \cdots v_{n-1}$$

Also, when $n \leqslant x \leqslant n+1$,

$$\int_n^x \phi(x)dx \geqslant - \int_n^x |\phi(x)|dx \geqslant - \int_n^{n+1} |\phi(x)|dx \geqslant -\mathrm{M},$$

where M is independent of x in virtue of (1). Adding the last two inequalities,

$$\int_m^x \phi(x)dx \geqslant \log \frac{u_n}{u_m} + v_m + v_{m+1} + \cdots v_n - \mathrm{M}$$

$$\geqslant \log \frac{u_n}{u_m} - \mathrm{S} - \mathrm{M} \text{ where } \mathrm{S} = \sum_m^\infty |v_n|$$

$$\therefore \quad e^{\int_m^x \phi(x)dx} \geqslant \frac{u_n}{u_m} e^{-\mathrm{S}-\mathrm{M}}$$

Integrating from n to $n+1$, we have

$$\int_n^{n+1} e^{\int_m^x \phi(x)dx} dx \geqslant \frac{u_n}{u_m} e^{-\mathrm{S}-\mathrm{M}}$$

The theorem now follows from a comparison of the given series with the series of positive terms $\sum^\infty \int_n^{n+1} e^{\int_m^x \phi(x)dx} dx$ which is, in fact, the integral $\int^\infty e^{\int_m^x \phi(x)dx} dx$

(For the divergence test, we start from $\phi(x) \leqslant \log r(n) + v_n$ and prove that $\int_n^{n+1} e^{\int_m^x \phi(x)dx} dx \leqslant \frac{u_n}{u_m} e^{\mathrm{S}+\mathrm{M}}$ whence the desired result follows as before)

Professor Brink's tests may be deduced from this theorem; for, if $0 < \mathrm{A} \leqslant r(x) \leqslant \mathrm{B}$, we may replace $\phi(x)$ by any continuous function of $r(x)$. If, further, $r(x)$ possesses a continuous derivative $r'(x)$ which is such that $\int^\infty |r'(x)|dx$ is convergent, we may replace $\phi(x)$

in the convergence test, by $[r(x)-1]$ . . . . . (C)

or by $\log r(x)$ . . . . . (C′)

in the divergence test, by $\left[1 - \frac{1}{r(x)}\right]$ . . . . . (D)

or by $\log r(x)$ . . . . . (D′)

Proof.—(C) We have to show that it is possible to choose v_n so that $\phi(x) - \log\ r(n) \geqslant v_n$. Now

$$\phi(x) - \log\ r(n) = r(x) - 1 - \log\ r(n) \geqslant r(x) - r(n)$$

since $r(n) - 1 \geqslant \log\ r(n)$ when $r(n) > 0$.

$$\therefore \quad \phi(x) - \log r(n) \geqslant \int_n^x r'(x)dx \geqslant -\int_n^x |r'(x)|\,dx \geqslant -\int_n^{n+1} |r'(x)|\,dx.$$

The last integral may therefore take the place of v_n since $\sum^{\infty} \int_n^{n+1} |r'(x)|\,dx$ is convergent.

(C′) In this case, $\phi(x) - \log\ r(n) = \log\ r(x) - \log\ r(n) = \int_n^x \frac{r'(x)}{r(x)}\,dx$

$$\geqslant -\int_n^x \frac{|r'(x)|}{|r(x)|}\,dx \geqslant -\frac{1}{A}\int_n^{n+1} |r'(x)|\,dx$$

Hence the last integral may replace v_n in the theorem.

(D) Here $\phi(x) - \log\ r(n) = 1 - \frac{1}{r(x)} - \log\ r(n) \leqslant \frac{1}{r(n)} - \frac{1}{r(x)}$

(since $1 - \frac{1}{r(n)} \leqslant \log r(n)$ when $r(n) > 0$,)

$$= \int_n^x \frac{r'(x)}{[r(x)]^2}\,dx$$

$$\leqslant \int_n^x \frac{|r'(x)|}{|r(x)|^2}\,dx$$

$$\leqslant \frac{1}{A^2}\int_n^{n+1} |r'(x)|\,dx$$

which may therefore be taken as the v_n of the theorem.

(D′) It can be proved, as in (C′), that

$$\phi(x) - \log\ r(n) \leqslant \frac{1}{A}\int_n^{n+1} |r'(x)|\,dx$$

It will be clear from this that our theorem includes Professor Brink's tests in much the same way as one of his own theorems in the *Annals of Mathematics* (Second Series, Volume 21, page 41) includes a test proposed by Hardy in the *Proceedings of the London Mathematical Society*. (Second Series, Volume 9, page 127).

The Christian College, Madras. — C. T. RAJAGOPAL.

An Application of a Simple Transformation

G. A. Linhart* has been successful in investigating data from widely dissimilar fields of science which behave according to the law

$$y = y_{\infty} kx^k/(1 + kx^{K}) \tag{1}$$

The constant K can be determined by applying the simple transformation

$$y = Y\ y'X, \qquad x = X \tag{2}$$

where y' is used for dy/dx.

After simplification equation (1) becomes:

$$Y = \frac{1}{K} + \frac{k}{K} X^{K} \tag{3}$$

so K is the reciprocal of the y-intercept of the transformed equation.

The transformation is easily applied graphically to the experimental data.

The transformed abscissa will be the same as the experimental abscissa. The new ordinate will be given by

$$Y = y/(y'X)$$

By sketching a tangent line to the curve of the experimental data and substituting the ratio of y to the subtangent for y', the value of Y is found to be the ratio of the subtangent to the abscissa of the point, with positive or negative sign according as the slope of the tangent line is positive or negative.

Then by erecting an ordinate equal to Y at the abscissa of the point in question, and repeating for several points near the y-axis we can get a good approximation to $\frac{1}{K}$ from the y-intercept of the transformed data.

Santa Monica,
CALIFORNIA.

L. J. ADAMS.

* G. A. Linhart, *Journal Phys. Chem.* Vol. XXXVII, No. 5, 1933
Journal Chem. Phys. Nov. 1933

On a Brocard—Ramanujan Problem

H. Brocard in 1876, and S. Ramanujan in 1913, asked for values of n which make $n!+1$, a square.

In 1885, Brocard suggested that the only solutions are 4, 5 and 7.

In 1906 A. Gerardin remarked that if

$$n!+1=m^2, \qquad \ldots(1)$$

has solutions other than $n=4, 5, 7$; $m=5, 11, 71$; then m has at least 20 digits.

I now find that (1) has no solutions in the range $7<n\leqslant 63$. This enables us to substitute 45 for 20 in Gerardin's remark. For the purpose of this verification $n!$ had actually to be calculated for $n\leqslant 63$.

Govt. College, *Hoshiarpur.* HANSRAJ GUPTA.

Decompositions into Cubes of Primes

Proceeding on lines similar to those indicated in my paper[1] entitled "Decompositions into Squares of Primes," I have now found that all positive integers $N\leqslant 10^5$, can be represented as the sums of at most 13 cubes of primes $\geqslant 1$. It is very probable that 1301 is the only number which requires as many as 13 cubes of primes for its representation.

Govt. College, *Hoshiarpur, 17-7-35.* HANSRAJ GUPTA.

[1] *Proc. Ind. Acad. Sc.*, I, (1935), No. 11.

38. On cross-examination the Plaintiff's attorney asked the husband :—

Is it not the case Mr. B———, that when a vehicle swerves sharply to the right the centrifugal force will throw any passenger seated on the left of the driver to the left ?

"That may be!" responded the defendant.

"And yet you say that your wife was thrown to her right, thereby causing you momentarily to lose control of the wheel?" pursued the lawyer.

The witness hesitated. Then having a deep sigh, he replied: Ah, but you don't know my wife. She always does things different to everybody else.

[Per T. S. VENKATARAMAN ESQ.] JOHN O' LONDONS WEEKLY, *October 6th, 1934.*

Note on Euler's Conjecture

I shall denote by E $(n, m,)$ Euler's conjecture that

$$x_1^n + x_2^n + \ldots + x_m^n = y^n$$

is impossible * for non-zero integers $x_1, \ldots x_m$, y when $n > 2$ and $m < n$. E $(n, 2)$ for $n > 2$ is Fermat's Last Theorem, which has been proved in a number of special cases. E $(n, 3)$ remains unproved to this day, even for $n = 4$.

Let us denote by A (n) the conjecture that

$$x^n + y^n = \mathrm{A} z^n \qquad (n > 4)$$

(where A is any positive integer) has at most one set of non-zero integral solutions. I show that

Theorem † *For Odd* $n \leqslant 5$, E $(n,3) \rightleftharpoons$ A(n)

The point of this theorem is that it might be possible to prove A (n) for given values of n by known methods used in treating Fermat's Last Theorem (the case A = 1). Thus a proof of A(5) would involve that of E (5, 3).

Proof : If A(n) is false then

$$x_1^n + y_1^n = \mathrm{A} z_1^n, \qquad x_2^n + y_2^n = \mathrm{A} z_2^n$$

so that, since n is odd,

$$(x_1 z_2)^n + (y_1 z_2)^n + (-x_2 z_1)^n + (-y_2 z_1)^n = 0,$$

which contradicts E $(n, 3.)$ If E $(n, 3)$ is false then,

$$x_1^n + x_2^n + x_3^n + x_4^n = 0\ ;$$

so that if we put $\qquad x_1^n + x_2^n = \mathrm{A} z^n$

then $\qquad (-x_3)^n + (-x_4)^n = \mathrm{A} z^n$

and these equations contradict A(n).

Waltair. S. CHOWLA,

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* Apart from trival solutions.

† In words, E$(n, 3)$ implies A (n), and conversely, so that the conjectures are completely equivalent.

SOLUTIONS TO QUESTIONS

Question 1634

(J. A. DEVASAGAYAM, M.A.)—The three conics having a common focus and inscribed in the three parallelograms formed by any three pairs of parallel straight lines have concurrent common chords.

Solution by A. Ranganatha Rao and S. Narayanan

It is known that the three external centres of similitude of three mutually intersecting circles taken two by two, are collinear. Reciprocating this theorem with regard to the radical centre of the three circles, we obtain the required result.

Question 1637

(A. A. KRISHNASWAMY AYYANGAR).—Two conics have double contact at U, W and V is the common pole of UW. ABC is a triangle inscribed in one conic and PQR is a triangle inscribed in the other such that P lies on BC, and AB, AC meet PQ, PR respectively on UW. Prove that the six points UVWAQR lie on a conic.

Solution by A. Ranganatha Rao

Projecting U, W into circular points, we have to prove on the projected plane the following theorem :—

" v is the common centre of two concentric circles S, S′. Triangles abc, pqr are inscribed in S, S′ respectively such that bc passes through p and pq, pr are respectively parallel to ab, ac. Prove that $aqvr$ are concylic."

Proof: Since ab, pq have the same perpendicular bisector, it can be easily shown that $aq = bp$. In a similar way, $ar = pc$. Since $\angle\, bac = \angle\, qpr$ or its supplement, $bc : qr$ = radius of S : radius of S′ $= va : vq$. Now, $aq \cdot vr + ar \cdot vq = (aq + ar)vq = (bp + pc)vq = bc \cdot vq = qr \cdot va$.

Hence, by the converse of Ptolemy's theorem, $aqvr$ are concyclic.

Question 1640

(C. N. SRINIVASIENGAR AND M. BHIMASENA RAO)

(i) Let $f(x,y,z)=0$ be the equation of any conic in areal co-ordinates. If $F(x,y,z)=0$ be the areal equation of any other curve, interpret the equation $F(XYZ)=0$ where $X=\frac{\partial f}{\partial y}-\frac{\partial f}{\partial z}$ etc.

(ii) Hence deduce that the eccentricity e of the general conic is given by

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(aa_0^2+bb_0^2+cc_0^2-2fb_0c_0\cos A_0-2gc_0a_0\cos B_0-2ha_0b_0\cos C_0)^2}{4\Delta^2(A+B+C+2F+2G+2H)}$$

where Δ is the area of the triangle of reference and a_0, b_0, c_0 are its sides.

(iii) If ϕ the angle between the equiconjugate diameters of the Steiner in- or circum ellipse of a triangle, whose Brocard angle is ω prove that $\sin\phi=\tan 60^\circ \tan\omega$.

(iv) Deduce from (1) the following generalisation of a result of Prof. M. T. Naraniengar (*Jour. Ind. Math. Soc.* Vol, XV. p. 51)

If $\phi(x,y,z)=0$ be the equation of a pair of conjugate diameters of the conic $f(x,y,z)=0$, $\phi(X,Y,Z)=0$ denotes the same lines.

Solution by Hukam Chand

(i) The equation $F(X,Y,Z)=0$ represents the polars with respect to $f(x,y,z)=0$ of the points at infinity on $F(x,y,z)=0$.

For let (x',y',z') be a point at infinity on $F(x,y,z)=0$; then the polar of this point w.r.t. $f(x,y,z)=0$ is

$$x'\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+z'\frac{\partial f}{\partial z}=0 \quad \ldots \quad (1)$$

with the conditions

$$F(x',y',z')=0 \quad \ldots \quad (2)$$

and

$$\sum x'=0 \quad \ldots \quad (3)$$

From (1) and (3) $x'/X=y'/Y=z'/Z$; consequently the points on the polars of points like (x',y',z') satisfy the equation $F(x,y,z)=0$.

(ii) If θ be the angle between the asymptotes, then θ is also the angle between the lines $f(x,y,-x-y)=0$ i.e. between

$$(a+c-2g)\,x^2+2\,(c+h-f-g)\,xy+(b+c-zf)\,y^2=0, \quad \ldots \quad (4)$$

and

$$(2-e^2)^2\,/\,(1-e^2)=-4\cot^2\theta$$

Now the angle interval α between the lines (l_1, m_1, n_1), (l_2, m_2, n_2) with respect to the absolute

$$\Omega(l,m,n) \equiv \sum a^2{}_0 l^2 - 2 \sum b_0 c_0 mn \cos A = 0$$

is given by the equation $\tan^2 \alpha = (\Omega_1\Omega_2 - \Omega_{12}{}^2)/\Omega_{12}{}^2$ where

$$\Omega(l_r, m_r, n_r) \equiv \Omega_r \text{ and } \Omega_{12} \equiv \tfrac{1}{2}\left(\sum l_1 \frac{\partial}{\partial l_2}\right)\Omega_2$$

For the lines (4), $\Omega_{12} = \sum a_0{}^2 a - 2 \sum f b_0 c_0 \cos A$

and $\Omega_1\Omega_2 - \Omega_{12}{}^2 = -16\Delta^2 \quad (\sum A + 2\sum F)$.

Consequently

$$\frac{(2-e^2)^2}{1-e^2} = \frac{[\sum a^2{}_0 a - 2 \sum f b_0 c_0 \cos A]^2}{4\Delta^2(\sum A + 2\sum F)}$$

(iii) If ϕ be the angle between the equi-conjugate diameters then $(2-e^2)^2/(1-e^2) = 4 \operatorname{cosec}^2 \phi$

For the Steiner in-ellipse $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$,

$$4 \operatorname{cosec}^2\phi = (2-e^2)^2/(1-e^2) = \frac{[\sum a_0{}^2 + 2\sum b_0 c_0 \cos A]^2}{48\ \Delta^2} = \frac{(a_0{}^2 + b_0{}^2 + c_0{}^2)^2}{12\ \Delta^2}$$

Consequently $\operatorname{cosec}\phi = \dfrac{a_0{}^2 + b_0{}^2 + c_0{}^2}{4\sqrt{3}\,\Delta} = \cot\ \omega\ \cot\ 60°.$

We get the same result if we take the Steiner circum-ellipse

$$yz + zx + xy = 0.$$

(iv) Let P and Q be the points where the conjugate diameters CP, CQ (C being the centre of the conic) meet the line at infinity so that CP and CQ are represented by $\phi(x, y, z) = 0$. Now the polars of P and Q are CQ and CP respectively which, by (i), are represented by $\phi(x, y, z) = 0$. Hence the generalization follows.

Also solved by the Proposers

REVIEWS

EUGEN JAHNKE AND FRITZE EMDE: *Funktionentafeln mit formeln und Kurven*, Zweite Auflage. Mit 171 Textfiguren.

S. 1-330 (i-xviii) B. G. Teubner (1933) R.M. 16.

This second edition of Jahnke and Emde's Tables of functions is more than double the size of the first and, while retaining all the characteristic features of the earlier edition, contains additional matter which greatly enhances the value of the work. The compilation of a set of mathematical tables can be undertaken with two different aims in view, that of technical application and of pure mathematical technique. While this work before us does not definitely fall into either of these categories, it combines in itself the virtues of being both a useful book and a beautiful one.

The two most striking features of the new edition are the solutions of cubic equations and the graphical representations of functions of complex variables. Only those who know the labour of numerical computation of the real and complex roots of cubic equations can appreciate the beauty of the graphical methods contained in this book and realise the immense advantages of a figure like that on p. 23. As for the graphical representations of functions of a complex variable the method of "relief" invented by Emde in the course of his lectures to students of Electrical Technology is extensively used in the book and shows abundantly the beauty and the power of such a method. One feels by looking at these relief pictures that the barrier of mathematical formalism which separated him from the intimate knowledge of the higher transcendental functions like the Elliptic and Riemann Zeta functions is completely removed, thus enabling an insight into the behaviour of these functions never dreamt of before. The reliefs and the contour lines of constant amplitude and constant modulus appear at first strange and weird but a little practice with the figures transforms this strangeness to a deep knowledge of the properties of the functions. As instances in point we may mention the graphical representation of the zeroes, poles and essential singularities of the higher transcendental functions.

This book is no doubt valuable to the pure mathematician, but it is to the theoretical physicist that it should appeal most by reason of the power of its graphical methods. It is a great pity that considerations of space and time did not permit the pictorial representation of many more functions, and we hope that when a third edition is called for it will be greatly augmented by these necessary additions and deserve the title "Anschauliche Funktionentheorie".

As regards accuracy, one certainly does not find here the standard set by the Milne Thomson-Comrie, and the authors are aware of this when they state in the preface that no attempt has been made to obtain more than a reasonable accuracy. It is no doubt true that absolute accuracy is an unattainable ideal, but still it would have been better if the work of the Mathematical Tables Committee of the British Association had been followed in greater detail than is done in the book.

The get-up of the book is excellent and is easily one of the finest examples of the art of printing. The bold type used in the tables is very pleasing to the eye and the care bestowed on the drawings is worthy of all praise. The price of 16 R.M. is quite reasonable and should go a long way in ensuring the success of a book which richly deserves it on the excellence of its contents. It is not often that a reviewer has the good fortune of recommending a book unreservedly and Jahnke and Emde's *Funktionentafeln* is certainly a rare good book.

K. S. K.

ISAAC MATHAI, M.A. (Hons.): *Matriculation Mathematical Tables* Mathai's Publications, Bombay, 74 pp., Price Re. 1.

The author who is a teacher in the Bombay Presidency believes that a familiarity with figures and their properties should help in conquering the distaste for and fear of mathematical studies on the part of students. The booklet under review contains besides numerical tables, a collection of formulae relating to arithmetic, algebra, mensuration, and trigonometry. Under the "general properties of figures" the author gives, among others, multiplication tables (30 × 12), primes under 1000; perfect numbers; saturated numbers (for which the number of divisors is greater than for any lower integer); squares, cubes, square roots, cube roots and reciprocals and rules of division. There are also tables of logarithms, anti-logarithms, sine, cosine and tangent tables, and some tables of physical constants at the end.

A. N. RAO.

BOOKS RECEIVED FOR REVIEW

WEBSTER-PLIMPTON: *Partial Differential Equations of Mathematical Physics*, B. G. Teubner, RM. 18.—1933.

JAHNKE-EMDE: *Functionentafeln mit Formeln und Kurven*: B. G. Teubner RM. 16.

SEIFERT-THRELFALL: *Lehrbuch der Topologie*, B. G. Teubner; RM. 20.

SMITH, David Eugene: *The Poetry of Mathematics and other essays*: Scripta Mathematica Library No. 1, 1934.

KEYSER: *Mathematics and the question of Cosmic Mind with other essays*, Scripta Mathematica Library No. 2; 1935.

Proceedings of the Second Karnataka Conference, Mysore, Price Re. 1.

RAY Upendranath: *Analytical Conics*; Published by the Author, 1928, Calcutta pp. 996.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the society:

S. N. Kaul Esq., M.A., Professor, S. P. College, Srinagar.

R. Kothandaraman Esq., B.A. (Hons.), Assistant, Dept. of Education, Health and Lands, New Delhi.

K. S. Puri Esq., Messrs. Puri Brothers, Puri Lodge, Patiala.

G. S. Marathey Esq., M.A., A.I.A. (Lond.), Actuary, Poona.

As already announced the next Conference of the Indian Mathematical Society will be held at Delhi between the 19th and 21st of December 1935.

The First Annual Meeting of the Indian Academy of Sciences, Bangalore, will be held at Bombay on Wednesday, 18th December 1935 and on succeeding days.

The Indian Science Congress will meet at Indore from the 2nd to the 8th of January 1935 under the patronage of H. H. Maharaja of Indore. The President Elect is Rai Sir Upendranath Brahmachari, Bahadur, KT., M.A., M.D., Ph.D. The sectional president for the Mathematics, Physics section will be Dr. T. Royds, D.Sc., Director of the Kodaikanal observatory. The Local Secretaries will be Dr. S S. Deshpande, Vice-Principal, Holkar College, Indore and Mr. K. A. Patwardhan, Daly College, Indore, to whom all enquiries should be addressed.

Prof. K. B. Madhava, of Mysore has been invited to deliver a course of six lectures on the Theory of Graduation under the auspices of the Indian Statistical Institute, Calcutta.

Dr. Chowla of the Andhra University will deliver a course of three University lectures on the Additive Theory of Numbers at Annamalainagar during the second week of November 1935.

The Calcutta Mathematical Society has announced the following subject for the first competition for the Krishna Kumari Ganesh-Prasad Prize and Medal: Lives and works of the ten famous Hindu Mathematicians: Aryabhatta, Varahamihira, Bhaskara I, Lalla, Brahmagupta, Sridhara, Mahavira, Sripati, Bhaskara II, Narayan.

The gold medal for the Royal Astronomical Society has been awarded to Prof. E. A. Milne, Rouse Ball Professor of Mathematics at the University of Oxford, for his work on radiative equilibrium and the theory of stellar atmospheres.

The staff of the School of Mathematics at the Institute for Advanced Study at Princton now consists of the following members: Drs.: Albert Einstein, Oswald Veblen, J. W. Alexander, John von Neumann, Herman Weyl and Marstan Morse.

A committee representing the departments of Mathematics and Philosophy of several American Universities has been appointed to provide greater facilities for the publication of papers in Symbolic Logic. It is proposed to start a Journal if there is sufficient support. Those who are willing to join the proposed Association with dues of probably $ 3·00 a year may communicate with Prof. C. J. Ducasse of Brown University.

The following exchanges have been effected:

Proceedings of the Research Institute for Mathematics and Mechanics, Tomsk.

Duke Mathematical Journal, a quarterly published by the Duke University from March 1935.

Sir Horace Lamb, well known as the Author of books on Mechanics and Hydrodynamics, died in December 1934.

QUESTIONS FOR SOLUTION

1707. (A. A. KRISHNASWAMY AYYANGAR).—(1) A, A′ denote two integers such that the order of the digits in one is reversed in the other. Investigate the existence of numbers which can be represented in more than one way as the product of two such numbers as A, A′.

$$Eg. \quad 3472 \times 2743 = 1456 \times 6541 = 9523696.$$

1708. (THÉBAULT, Le Mans).—With the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 taken each once, form the least and biggest numbers which are perfect squares.

1709. (R. S. VAIDISWARAN).—There are three colours and n balls of each colour. In how many ways can they be arranged in three dissimilar bags, each containing n.

1710. (V. RAMASWAMI AIYAR.).—Let ABCD be a quadrangle and M its *Durairajan point*. Let α, β, γ, α', β', γ' be the feet of the perpendiculars from M on BC, CA, AB, DA, DB, DC. Prove the following:

(1) $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ bisect one another at a point O;

(2) The points α, β, γ, α', β', γ' and M lie on a conic centre O;

(3) The pedal circles of M w. r. t. triangles BCD, CDA, DAB ABC concur at μ, the centre of the rectangular hyperbola ABCD;

(4) The diameters of these pedal circles are equal to MA, MB, MC, MD;

(5) The point μ lies on the conic in (2) diametrically opposite to M.

1711. (V. RAMASWAMI AIYAR.).—Let ABCD be a given quadrangle and μ a given point. Consider the totality of points P whose pedal circles w. r. t. triangles BCD, CDA, DAB, ABC concur at μ:

(1) If μ be the centre of the rectangular hyperbola ABCD, show that the tolality of points P consists of the *Durairajan point* of the quadrangle together with all points lying on the rectangular hyperbola ABCD.

(2) If μ be any other point (finite point), the points P are three in number. Explain this discontinuous feature.

(3) In case (2), find intrinsic relations between the P's by which if *one* of them be assigned, we can fix (*visualize*, so to say) the other two.

1712. (A. RANGANATHA RAO.).—The normals at four points ABCD on a conic S are concurrent at O. Prove that the directrix of the parabola which touches the tangents to S at ABCD passes through O and that the parabola is inpolar to S.

1713. (FRANCIS REGAN, St. Louis, U. S. A.).—Prove that if through any fixed point V on a conic, pairs of lines are drawn, making supplementary angles with any fixed line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings both national and sectional, are reported in the official Journal.

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Remittances of membership fees and of subscriptions should be sent to the Treasurer, Prof. L. N. SUBRAMANIAM, Department of Mathematics, Christian College, Madras.